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Statistical thermodynamics of superfluid helium confined to a cuboidal enclosure below 0·5 K†

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Abstract. We have calculated the low-temperature specific heat and normal fluid density of superfluid helium confined to a completely general cuboidal geometry ($L_1 \times L_2 \times L_3$). The limiting cases of a film and of a capillary geometry can be obtained readily and are found to be in agreement with the ones recently examined by Pajkowski and Pathria. New results on the cubic geometry are presented here for the first time. A systematic application of the Poisson summation formula reveals the manner in which the magnitude of the finite-size corrections, in the asymptotic limit ($L/\lambda \gg 1$), is directly linked with the dimensionality of the system.

1. Introduction

Recent interest in finite-size effects has prompted numerous experimental and theoretical investigations in various physical systems, notably in superfluid helium. Experimental studies on this system in thin films, narrow channels and porous matrices (Rudnick *et al* 1967, 1968, Fraser and Rudnick 1968, Kagiwada *et al* 1969, Henkel *et al* 1969, Pobell *et al* 1972, Scott *et al* 1972, Gregory and Lim 1974) have indicated that the various low-temperature properties deviate significantly from their values in the bulk situation. In an attempt to provide a theoretical understanding of these deviations, Padmore (1972) suggested that the finite-size effects in superfluid helium may be analysed within the framework of Landau's quasiparticle picture, provided that the smallest dimension of the enclosure is much larger than the healing length, which is of the order of 1–2 Å. For $T < 0{\cdot}5$ K, the contribution of rotons to the various thermodynamic properties of the system is negligible, so one may consider the phonon contribution alone. In this spirit, Padmore (1972) and subsequently others (Haug 1973, Pajkowski and Pathria 1978, 1979) have examined the thermodynamics of a phonon gas confined to the geometry of films and channels and subjected to various boundary conditions. In this paper we generalise that work to an arbitrary cuboidal geometry ($L_1 \times L_2 \times L_3$) under periodic boundary conditions and show how the results obtained by the previous authors are recovered when one or two dimensions of the enclosure become infinite. The case of a cubic geometry, which is completely finite, is also discussed at some length. Our analysis automatically reveals how, in the asymptotic limit ($L/\lambda \gg 1$), the magnitude of the finite-size corrections is affected by the dimensionality of the system; here, $\lambda (= \hbar c/kT)$ is the mean thermal wavelength of the phonons.

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In this study we concentrate on the normal fluid density ρ_n, given by

$$\rho_n(T) = \frac{\hbar^2}{kTV} \sum_k k_z^2 \, e^{\beta\epsilon_k} (e^{\beta\epsilon_k} - 1)^{-2} \tag{1}$$

and the specific heat C_V, given by

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V, \qquad U = \sum_k \epsilon_k (e^{\beta\epsilon_k} - 1)^{-1}. \tag{2}$$

Here

$$\epsilon_k = \hbar c (k_1^2 + k_2^2 + k_3^2)^{1/2}, \qquad k_i = \frac{2\pi}{L_i} n_i \qquad (n_i = 0, \pm 1, \pm 2, \ldots). \tag{3}$$

2. The cuboidal geometry

The major task in the study of finite systems lies in evaluating the discrete sums-over-states such as the ones appearing in equations (1) and (2). The method highly suited for this purpose is the application of the Poisson summation formula (PSF); see, for instance, Schwartz (1966).

In (1) the summations over n_1 and n_2 are on a different footing from the one over n_3; we therefore apply the two-dimensional PSF to the sums over n_1 and n_2, and subsequently apply the one-dimensional PSF to the sum over n_3. The first step results in

$$\rho_n = \frac{\hbar}{Vc\lambda} + \frac{16\pi^3\hbar}{c}\left(\frac{\lambda}{L_3}\right) \sum_{q_{1,2}=-\infty}^{\infty} \sum_{n_3=1}^{\infty} \left(\frac{n_3}{L_3}\right)^2 \times \sum_{j=1}^{\infty} j \int_0^{\infty} \exp[-2\pi\lambda j (y^2 + n_3^2/L_3^2)^{1/2}] J_0(2\pi q' y) y \, dy, \tag{4}$$

where $J_0(z)$ is the Bessel function of the first kind and $q' = (q_1^2 + q_2^2)^{1/2}$. Performing the integration over y and applying the PSF to the sum over n_3 then leads to

$$\rho_n = \frac{\hbar}{Vc\lambda} + \frac{4\hbar}{\pi^2 c\lambda^4} \sum_{q_{1,2,3}=-\infty}^{\infty} \sum_{j=1}^{\infty} j^2 \left(\frac{1}{[j^2 + (q''/\lambda)^2]^3} - \frac{6(L_3 q_3/\lambda)^2}{[j^2 + (q''/\lambda)^2]^4}\right), \tag{5}$$

where $q'' = (q_1^2 + q_2^2 + q_3^2)^{1/2}$. We note that the term with $q = 0$ gives the standard bulk result $\rho_n^{\infty} (= 2\pi^2\hbar/45c\lambda^4)$, while the remaining terms represent finite-size corrections. For studying special cases it will be convenient to rewrite this result in terms of hyperbolic functions which appear on carrying out the summation over j, namely

$$\rho_n = \rho_n^{\infty} + \frac{\hbar}{Vc\lambda} + \frac{\hbar}{2\pi^2 c\lambda^4} \sum_{q_{1,2,3}}^{\infty}{}' = -\infty \times \left[\!\!\left[\left[\frac{\pi}{2}\left(\frac{\lambda}{q''}\right)^3 \coth\left(\frac{\pi q''}{\lambda}\right) + \frac{\pi^2}{2}\left(\frac{\lambda}{q''}\right)^2 \operatorname{cosech}^2\left(\frac{\pi q''}{\lambda}\right) - \pi^3\left(\frac{\lambda}{q''}\right) \coth\left(\frac{\pi q''}{\lambda}\right) \operatorname{cosech}^2\left(\frac{\pi q''}{\lambda}\right)\right]\right.\right.$$

$$-\left(\frac{L_3q_3}{\lambda}\right)^2\left\{\frac{3\pi}{2}\left(\frac{\lambda}{q''}\right)^5\coth\left(\frac{\pi q''}{\lambda}\right)+\frac{3\pi^2}{2}\left(\frac{\lambda}{q''}\right)^4\operatorname{cosech}^2\left(\frac{\pi q''}{\lambda}\right)\right.$$

$$\left.\left.\left.-\pi^4\left(\frac{\lambda}{q''}\right)^2\left[\operatorname{cosech}^4\left(\frac{\pi q''}{\lambda}\right)+2\coth^2\left(\frac{\pi q''}{\lambda}\right)\operatorname{cosech}^2\left(\frac{\pi q''}{\lambda}\right)\right]\right\}\right]\right]. \tag{6}$$

Note that the primed summation in (6) does not include the term with $\boldsymbol{q}=0$.

The internal energy U is comparatively easier to handle owing to the symmetry among different variables. An application of the three-dimensional PSF now gives

$$\frac{\beta U}{V}=2\pi\lambda\sum_{q_{1,2,3}=-\infty}^{\infty}\int_0^\infty\frac{y^3}{(e^{2\pi\lambda y}-1)}\int_0^{2\pi}\int_0^{\pi}e^{-2\pi i q'' y\cos\theta}\sin\theta\,d\theta\,d\phi\,dy$$

$$=\frac{\beta U^\infty}{V}+\frac{1}{2\pi^2\lambda^3}\sum_{q_{1,2,3}=-\infty}^{\infty}{}'\left[\left(\frac{\lambda}{q''}\right)^4-\pi^3\left(\frac{\lambda}{q''}\right)\coth\left(\frac{\pi q''}{\lambda}\right)\operatorname{cosech}^2\left(\frac{\pi q''}{\lambda}\right)\right], \tag{7}$$

where $U^\infty=\pi^2V\hbar c/30\lambda^4$. The specific heat then follows after a straightforward differentiation.

3. Special cases

Equations (6) and (7) for the general cuboidal geometry form our starting point for a detailed examination of the various special cases. In the film geometry we let L_2, $L_3\to\infty$ and set $L_1=L$. The only values of q_2 and q_3 which contribute in this case are $q_2=q_3=0$. We are therefore left with one-dimensional sums alone:

$$\rho_n=\rho_n^\infty+\frac{\hbar}{\pi^2c\lambda^4}\sum_{q=1}^{\infty}\left[\frac{\pi}{2}\left(\frac{\lambda}{L}\right)^3\frac{\coth(\pi Lq/\lambda)}{q^3}+\frac{\pi^2}{2}\left(\frac{\lambda}{L}\right)^2\frac{\operatorname{cosech}^2(\pi Lq/\lambda)}{q^2}\right.$$

$$\left.-\pi^3\left(\frac{\lambda}{L}\right)\frac{\coth(\pi Lq/\lambda)\operatorname{cosech}^2(\pi Lq/\lambda)}{q}\right] \tag{8}$$

and

$$\frac{\beta U}{V}=\frac{\beta U^\infty}{V}+\frac{1}{\pi^2}\frac{\lambda}{L^4}\zeta(4)-\frac{\pi}{\lambda^2L}\sum_{q=1}^{\infty}\frac{\coth(\pi Lq/\lambda)\operatorname{cosech}^2(\pi Lq/\lambda)}{q}. \tag{9}$$

In the asymptotic regime ($L/\lambda\gg1$) the foregoing summations are strongly convergent. One may therefore retain only the leading exponential terms of the series and obtain for the normal fluid density

$$\rho_n=\frac{2\pi^2\hbar}{45c\lambda^4}\left[\!\left[1+\frac{135}{4\pi^3}\left(\frac{\lambda}{L}\right)\left\{\frac{\zeta(3)}{3}\left(\frac{\lambda}{L}\right)^2-\frac{8\pi^2}{3}\left[1-\left(\frac{\lambda}{2\pi L}\right)-\left(\frac{\lambda}{2\pi L}\right)^2\right]e^{-2\pi L/\lambda}\right\}\right]\!\right] \tag{10}$$

and for the specific heat per unit volume

$$c_v=\frac{2\pi^2k}{15\lambda^3}\left\{1+60\left[1-3\left(\frac{\lambda}{2\pi L}\right)\right]e^{-2\pi L/\lambda}\right\}, \tag{11}$$

in complete agreement with the corresponding results of Pajkowski and Pathria (1978).

In the complementary regime ($L/\lambda\ll1$) expansions (8) and (9) as such are not very useful, so we re-apply the one-dimensional PSF to them and obtain alternative expansions, namely

$$\rho_n=\frac{3\hbar}{2\pi c\lambda^3 L}\sum_{j=1}^{\infty}\left[\frac{\coth(\pi\lambda j/L)}{j^3}+\left(\frac{\pi\lambda}{L}\right)\frac{\operatorname{cosech}^2(\pi\lambda j/L)}{j^2}+\frac{2}{3}\left(\frac{\pi\lambda}{L}\right)^2\frac{\coth(\pi\lambda j/L)\operatorname{cosech}^2(\pi\lambda j/L)}{j}\right] \tag{12}$$

and

$$\frac{\beta U}{V}=\frac{1}{\pi\lambda^2 L}\sum_{j=1}^{\infty}\left[\frac{\coth(\pi\lambda j/L)}{j^3}+\left(\frac{\pi\lambda}{L}\right)\frac{\operatorname{cosech}^2(\pi\lambda j/L)}{j^2}+\left(\frac{\pi\lambda}{L}\right)^2\frac{\coth(\pi\lambda j/L)\operatorname{cosech}^2(\pi\lambda j/L)}{j}\right]. \tag{13}$$

Expressions (12) and (13) are strongly convergent for $L/\lambda \ll 1$, and, retaining only the leading exponential terms here, we obtain

$$\rho_n=\frac{3\hbar}{2\pi c\lambda^3 L}\left\{\zeta(3)+2\left[1+\left(\frac{2\pi\lambda}{L}\right)+\frac{1}{3}\left(\frac{2\pi\lambda}{L}\right)^2\right]e^{-2\pi\lambda/L}\right\} \tag{14}$$

and

$$c_v=\frac{3k}{\pi\lambda^2 L}\left\{\zeta(3)+\left[2+2\left(\frac{2\pi\lambda}{L}\right)+\left(\frac{2\pi\lambda}{L}\right)^2+\frac{1}{3}\left(\frac{2\pi\lambda}{L}\right)^3\right]e^{-2\pi\lambda/L}\right\}, \tag{15}$$

again in agreement with Pajkowski and Pathria (1978). Although in principle the pairs of expressions (10), (14) and (11), (15) are supposed to cover the extreme ranges of values of the parameter L/λ, in practice they have an overlapping range of utility, so that, taken together, they constitute a description of ρ_n and c_v over the whole range of film thickness. It should also be noted that both pairs of equations (8), (12) and (9), (13) are in fact valid over the entire range of L/λ, though they differ markedly as regards convergence. For instance, although (12) and (13) are strongly convergent for $L/\lambda \ll 1$, we can recover even the bulk result from them by using appropriate approximations for the hyperbolic functions.

For the square-channel geometry we let $L_3 \to \infty$ and set $L_1 = L_2 = L$ in equations (6) and (7). The only value of q_3 contributing in this case is $q_3 = 0$, so we are left with two-dimensional sums, namely

$$\rho_n=\rho_n^{\infty}+\frac{\hbar}{2\pi^2 c\lambda^4}\sum_{q_{1,2}=-\infty}^{\infty}{}'\left[\frac{\pi}{2}\left(\frac{\lambda}{L}\right)^3\frac{\coth(\pi Lq/\lambda)}{q^3}+\frac{\pi^2}{2}\left(\frac{\lambda}{L}\right)^2\frac{\operatorname{cosech}^2(\pi Lq/\lambda)}{q^2}-\pi^3\left(\frac{\lambda}{L}\right)\frac{\coth(\pi Lq/\lambda)\operatorname{cosech}^2(\pi Lq/\lambda)}{q}\right] \tag{16}$$

and

$$\frac{\beta U}{V}=\frac{\beta U^{\infty}}{V}+\frac{2\lambda}{\pi^2 L^4}\zeta(2)\beta(2)-\frac{\pi}{2\lambda^2 L}\sum_{q_{1,2}=-\infty}^{\infty}{}'\frac{\coth(\pi Lq/\lambda)\operatorname{cosech}^2(\pi Lq/\lambda)}{q}, \tag{17}$$

where we have used the Hardy sums

$$\theta(s)=\sum_{q_{1,2}=-\infty}^{\infty}{}'\frac{1}{(q_1^2+q_2^2)^s}=4\zeta(s)\beta(s), \tag{18}$$

$\beta(s)$ being the Dirichlet L-series $\Sigma_{n=0}^{\infty}(-1)^n(2n+1)^{-s}$.

In the asymptotic regime ($L/\lambda \gg 1$) these expansions are strongly convergent, so we may retain only the largest set of terms in each summation and obtain

$$\rho_n = \frac{2\pi^2\hbar}{45c\lambda^4}\left[\left[1+\frac{45}{2\pi^3}\left(\frac{\lambda}{L}\right)\left\{\frac{\theta(3/2)}{4}\left(\frac{\lambda}{L}\right)^2 - 8\pi^2\left[1-\left(\frac{\lambda}{2\pi L}\right)-\left(\frac{\lambda}{2\pi L}\right)^2\right]e^{-2\pi L/\lambda}\right\}\right]\right] \tag{19}$$

and

$$c_v = \frac{2\pi^2 k}{15\lambda^3}\left\{1+120\left[1-3\left(\frac{\lambda}{2\pi L}\right)\right]e^{-2\pi L/\lambda}\right\}, \tag{20}$$

in complete agreement with the corresponding results of Pajkowski and Pathria (1979).

Again, expansions (16) and (17) do not provide a useful description in the complementary regime ($L/\lambda \ll 1$). For this we may transform these expressions by employing the basic identities

$$\frac{\pi}{2}\coth(\pi a) - \frac{1}{2a} = \sum_{n=1}^{\infty}\frac{a}{n^2+a^2}, \tag{21}$$

$$\sum_{q_{1,2}=-\infty}^{\infty}{}' \left(\frac{q}{\mu}\right)K_1(\mu q) = \frac{4\pi}{\mu^4} - \frac{1}{\mu^2} + 4\pi \sum_{q_{1,2}=-\infty}^{\infty}{}' \frac{1}{(\mu^2+4\pi^2 q^2)^2}, \tag{22}$$

and their relevant derivatives, to obtain

$$\rho_n = \frac{\pi\hbar}{3c\lambda^2L^2}\left\{1+\frac{6}{\pi^2}\sum_{j=1}^{\infty}\sum_{q_{1,2}=-\infty}^{\infty}{}' \left[\left(\frac{2\pi\lambda}{L}\right)\frac{q}{j}K_1\left(\frac{2\pi\lambda}{L}jq\right)+\frac{1}{2}\left(\frac{2\pi\lambda}{L}\right)^2 q^2 K_0\left(\frac{2\pi\lambda}{L}jq\right)\right]\right\} \tag{23}$$

and

$$c_v = \frac{\pi k}{3\lambda L^2}\left\{1+\frac{6}{\pi^2}\sum_{j=1}^{\infty}\sum_{q_{1,2}=-\infty}^{\infty}{}' \left[\left(\frac{2\pi\lambda}{L}\right)\frac{q}{j}K_1\left(\frac{2\pi\lambda}{L}jq\right)+\frac{1}{2}\left(\frac{2\pi\lambda}{L}\right)^2 q^2 K_0\left(\frac{2\pi\lambda}{L}jq\right)\right.\right.$$
$$\left.\left.+\frac{1}{2}\left(\frac{2\pi\lambda}{L}\right)^3 q^3 j K_1\left(\frac{2\pi\lambda}{L}jq\right)\right]\right\}. \tag{24}$$

Owing to the rapid convergence of the modified Bessel functions appearing in these expressions, they are highly useful in the regime $L/\lambda \ll 1$. Taking only the leading set of terms in each sum, and employing the well-known expansion

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2}\exp(-z)\left[1+\frac{4\nu^2-1}{8z}+\frac{(4\nu^2-1)(4\nu^2-9)}{2!(8z)^2}+\ldots\right], \tag{25}$$

we get

$$\rho_n = \frac{\pi\hbar}{3c\lambda^2L^2}\left\{1+24\left(\frac{\lambda}{L}\right)^{3/2}\left[1+\frac{15}{16}\left(\frac{L}{\pi\lambda}\right)+\frac{105}{512}\left(\frac{L}{\pi\lambda}\right)^2-\frac{315}{8192}\left(\frac{L}{\pi\lambda}\right)^3\right]e^{-2\pi\lambda/L}\right\} \tag{26}$$

and

$$c_v = \frac{\pi k}{3\lambda L^2}\left\{1+48\pi\left(\frac{\lambda}{L}\right)^{5/2}\left[1+\frac{11}{16}\left(\frac{L}{\pi\lambda}\right)+\frac{225}{512}\left(\frac{L}{\pi\lambda}\right)^2\right.\right.$$
$$\left.\left.+\frac{105}{1024}\left(\frac{L}{\pi\lambda}\right)^3-\frac{525}{8192}\left(\frac{L}{\pi\lambda}\right)^4\right]e^{-2\pi\lambda/L}\right\}. \tag{27}$$

Equations (26) and (27) are again in agreement with the corresponding ones of Pajkowski and Pathria (1979) and, taken together with equations (19) and (20), constitute a complete description of ρ_n and c_v over the entire range of L/λ. It may be mentioned here too that, although expressions (23) and (24) are useful in the regime $L/\lambda \ll 1$, they are valid over the entire range of that parameter, as can be seen by integrating over $\boldsymbol{q}$ (which corresponds to taking the bulk limit, and indeed leads to the usual bulk results).

We finally consider the case of a cube, for which $L_1 = L_2 = L_3 = L$. Equation (6) now becomes

$$\rho_n = \rho_n^\infty + \frac{\hbar}{c\lambda L^3} + \frac{\hbar}{2\pi^2 c\lambda^4} \sum_{q_{1,2,3}=-\infty}^{\infty}{}' \left\{ \frac{\pi^4}{3} \left[\operatorname{cosech}^4\left(\frac{\pi L}{\lambda} q\right) + 2\coth^2\left(\frac{\pi L}{\lambda} q\right) \operatorname{cosech}^2\left(\frac{\pi L}{\lambda} q\right) \right] - \pi^3 \left(\frac{\lambda}{L}\right) \frac{\coth(\pi L q/\lambda)\operatorname{cosech}^2(\pi L q/\lambda)}{q} \right\}, \tag{28}$$

where we have symmetrised the sum involving q_3 with respect to all three indices. One cautionary remark is in order here: in view of the hydrodynamical basis of defining ρ_n, as in equation (1), calculations on a completely finite geometry might be questionable if one chose boundary conditions which do not permit flow (Dirichlet, for example). Since our treatment only deals with periodic boundary conditions, this objection does not apply here.

Equation (7) for this case takes the form

$$\frac{\beta U}{V} = \frac{\beta U^\infty}{V} + \frac{\lambda}{2\pi^2 L^4} b_2 - \frac{\pi}{2\lambda^2 L} \sum_{q_{1,2,3}=-\infty}^{\infty}{}' \frac{\coth(\pi L q/\lambda)\operatorname{cosech}^2(\pi L q/\lambda)}{q}, \tag{29}$$

where (Zasada and Pathria 1976)

$$b_2 = \sum_{q_{1,2,3}=-\infty}^{\infty}{}' \frac{1}{(q_1^2 + q_2^2 + q_3^2)^2} \simeq 16.53232. \tag{30}$$

As before, retaining only leading terms in these summations, we obtain expressions appropriate to the asymptotic régime ($L/\lambda \gg 1$):

$$\rho_n = \frac{2\pi^2 \hbar}{45 c\lambda^4} \left\{ 1 + \frac{45}{2\pi^2}\left(\frac{\lambda}{L}\right)^3 + 180\left[1 - 3\left(\frac{\lambda}{2\pi L}\right)\right] e^{-2\pi L/\lambda} \right\} \tag{31}$$

and

$$c_v = \frac{2\pi^2 k}{15\lambda^3} \left\{ 1 + 180\left[1 - 3\left(\frac{\lambda}{2\pi L}\right)\right] e^{-2\pi L/\lambda} \right\}. \tag{32}$$

We note that the finite-size correction to c_v in the case of a cube, in the asymptotic régime, has the same functional dependence on L/λ as in the case of films and channels; see equations (11), (20) and (32). However, the magnitudes of enhancement over the bulk value in the three cases are in the ratios 1 : 2 : 3. Although it is intuitively clear that the finite-size corrections should be larger in a system with a greater degree of finiteness, here we see rather explicitly that this effect, in the asymptotic régime, is directly linked with the dimensionality of the system through the multiplicity of the leading terms in the relevant sums. We observe that these corrections are given by one-, two-, and three-dimensional summations over the same summand (see equation (7)) for all geometries, and the leading terms are two-, four- and six-fold degenerate, thus accounting for the ratios of the factors obtained. The corresponding correction to ρ_n, however, exhibits somewhat different behaviour for different geometries. This is not surprising because a hydrodynamical property, with its inherent anisotropy, such as in (1), is quite likely to possess a sensitivity which changes more characteristically with geometry.

Returning to the cube, we shall now obtain expressions useful in the complementary régime ($L/\lambda \ll 1$). For this it is sufficient to retain only the leading terms in the original relations, equations (1) and (2), for in this régime those relations are already strongly convergent. We obtain

$$\rho_n = \frac{\hbar}{c\lambda L^3}\left\{1 + \left(\frac{2\pi\lambda}{L}\right)^2 \sum_{j=1}^{\infty} j \sum_{n_{1,2,3}=-\infty}^{\infty}{}' n_3^2 \exp\left[-\left(\frac{2\pi\lambda}{L}\right) j (n_1^2+n_2^2+n_3^2)^{1/2}\right]\right\} \tag{33}$$

and

$$\beta U = \left\{1 + \left(\frac{2\pi\lambda}{L}\right) \sum_{j=1}^{\infty} \sum_{n_{1,2,3}=-\infty}^{\infty}{}' (n_1^2+n_2^2+n_3^2)^{1/2} \exp\left[-\left(\frac{2\pi\lambda}{L}\right) j (n_1^2+n_2^2+n_3^2)^{1/2}\right]\right\}. \tag{34}$$

Retaining only the largest leading terms, we obtain

$$\rho_n = \frac{\hbar}{c\lambda L^3}\left[1 + 2\left(\frac{2\pi\lambda}{L}\right)^2 e^{-2\pi\lambda/L}\right] \tag{35}$$

and

$$c_v = \frac{k}{L^3}\left[1 + 6\left(\frac{2\pi\lambda}{L}\right)^2 e^{-2\pi\lambda/L}\right]. \tag{36}$$

The expressions in this régime vary rather significantly in their functional form as we go from (i) a film to (ii) a channel to (iii) a cube; cf equations (14), (15), equations (26), (27) and equations (35), (36). Since $L \ll \lambda$, the finite dimensions practically drop out, and the system under study behaves as if it were essentially (i) a two-dimensional or (ii) a one-dimensional or (iii) a 'zero-dimensional' bulk system.

Once again the approximations (31) and (32), coupled with the approximations (35) and (36), cover almost the entire range of λ/L; see figures 1 and 2, where results for other geometries are also plotted.

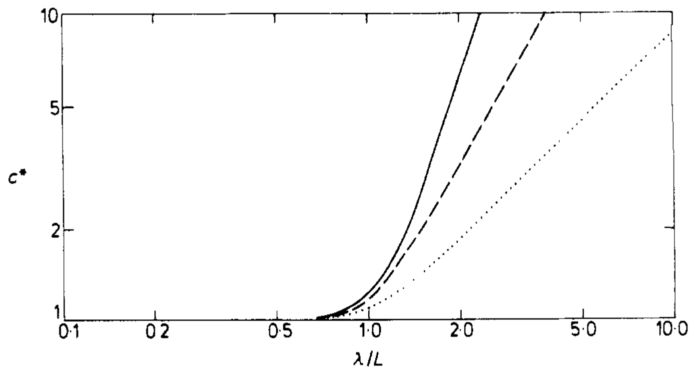

Figure 1. Logarithmic plot of the reduced specific heat $c^*(=c_v/c_v^\infty)$ of a phonon gas for various goemetries under periodic boundary conditions as a function of λ/L. Cube: full curve; square channel: broken curve; film: dotted curve.

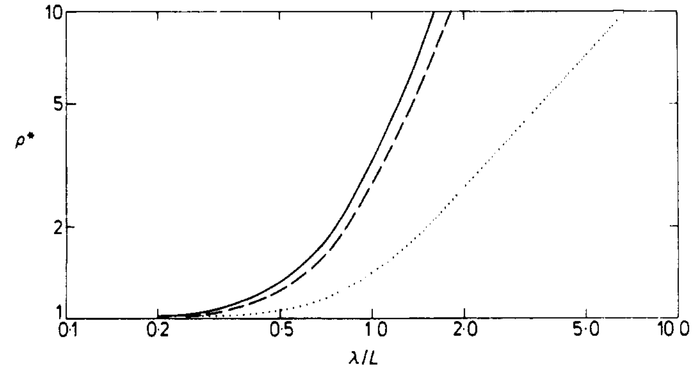

Figure 2. Logarithmic plot of the reduced normal fluid density $\rho^*(=\rho_n/\rho_n^\infty)$ of a phonon gas for various geometries under periodic boundary conditions as a function of λ/L. Details as in figure 1.

4. Concluding remarks

Examining figures 1 and 2, we notice all the qualitative features one would expect of the quantities $c^*(=c_v/c_v^\infty)$ and $\rho^*(=\rho_n/\rho_n^\infty)$ as functions of λ/L. In each case the system exhibits a crossover from bulk to finite-size behaviour when $\lambda/L = 0(1)$, i.e. when the mean thermal wavelength of the phonons is of the same order of magnitude as the finite dimension of the container. Quite expectedly, the crossover occurs faster for a more finite geometry, with deviations from bulk behaviour extending progressively deeper into the asymptotic régime.

Although there have been some measurements of the superfluid density ρ_s in liquid helium confined to restricted geometries below 0·5 K (Pobell *et al* 1972) which agree qualitatively with the calculations reported here, in that the superfluid density is suppressed below its bulk value (indicating an enhancement of the normal fluid density ρ_n), a meaningful comparison on a quantitative basis is not yet possible. The fourth-sound, oscillating U-tube and persistent current experiments of these authors were conducted in compressed lampblack or Vycor in which a geometry of non-intersecting channels was assumed. However, nitrogen absorption and mercury intrusion methods, in conjunction with the above assumption, gave substantially different values for the average pore diameter in these matrices; this makes the numerical value of λ/L very uncertain. Hopefully, when precise methods for a quantitative determination of the interstitial geometry in such structures are available, a detailed comparison with theoretical results will be possible. In the meantime, the calculations reported here might stimulate further experimental work on the specific heat c_v and the superfluid density ρ_s of liquid helium confined to restricted geometries in the phonon régime ($T < 0{\cdot}5$ K).

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